Cocycles, radicals and splitting fields of twisted group algebras

Hans Opolka

TU Braunschweig

Universitätsplatz 2

D-38106 Braunschweig

e-mail: h.opolka@tu-bs.de

Abstract: For a field k of characteristic 0, a finite group G and a central 2-cocycle $f: G \times G \to k^*$ denote by (k, G, f) the corresponding twisted group algebra. The purpose of this note is to show that a certain radical extension of k, which is constructed from f, is a splitting field of (k, G, f), and to illustrate this result by an example which is related to Gauss sums.

Key words: Semisimple algebras, splitting fields

MSC classification 2010: 16K20, 16S35, 20C25

For basic definitions and results about group cohomology and twisted group algebras resp. about general finite dimensional associative algebras which will be used in this note see e.g. [Y], [CO] resp. [BR], [CR]. For the case of the ordinary group algebra $f = 1$ we also refer to [SE] and [LA], XVIII.

Let k be a field of characteristic 0, let C be an algebraic closure of k, let G be a finite group, let $f: G \times G \to k^*$ be a central 2-cocycle and denote by (k, G, f) the corresponding twisted group algebra. As is well known it is semisimple. Consider the function $a_f: G \to k^*$ defined by

$$a_f(x) := \prod_{i=1}^{m(x)} f(x, x^i), \ x \in G,$$

where $m(x)$ denotes the order of an element $x \in G$. For every $x \in G$ fix the set of all roots $\alpha_f(x) \in C$ of $a_{f(x)}$ of order dividing $m(x)$and denote by L/k the subextension of C/k which is obtained from k by adjoining to k all $\alpha_f(x), x \in G$. The main result of this note is as follows

(1) Theorem *The field* L *is a splitting field of* (k, G, f)

We note that this result, applied to $k = \mathbb{Q}$ and $f = 1$, contains the well known result of R. Brauer, see e.g. [SE], 12.3 or [LA], XVIII, §11, Thm. 17, according to which the field extension of $\mathbb{Q}$ which is obtained from $\mathbb{Q}$ by adjoining to $\mathbb{Q}$ a primitive m-th root of unity

https://doi.org/10.24355/dbbs.084-201810301515-0

$\zeta \in \mathbb{C}$, where m is the exponent of G, is a splitting field of the ordinary group algebra $\mathbb{Q}[G] = (\mathbb{Q}, G, 1)$.

In the proof of **(1)** we will use some results on representations of twisted group algebras. The first is the twisted version of the reciprocity law (1.12) in [O1] in terms of characters: Let $f, f' : G \times G \to k^*$ be central 2-cocycles, let $H \leq G$ be a subgroup, let χ be a $k-f-$character of G, i.e. the character of a representation of (k, G, f), viewed as a function $\chi: G \to k$, and let γ be a $k - f'-$character of H. Then

(2) $\chi \cdot Ind_H^G(\gamma)_{f'} = Ind_H^G(\chi_{|H} \cdot \gamma)_{f \cdot f'}$.

Here $Ind_H^G(\theta)_t$ denotes the $k - t-$ character of G which is induced by the $k - t-$character θ of H with respect to the central 2-cocycle $t: G \times G \to k^*$, i.e.

$$Ind_H^G(\theta)_t(x) = \frac{1}{|H|} \cdot \sum_{g \in G} \frac{t(g,x)t(gx,g^{-1})}{t(g,g^{-1})} \theta^*(gxg^{-1}) \,,\; x \in G,$$

where $\theta^*(z) := \theta(z)$ for $z \in H$ and $\theta^*(z) \coloneqq 0$ for $z \in G - H$.

Denote by W_k the group of roots of unity in k. It is well known, see e.g. $[O1], (1.2)$ and (1.4), that there is a subgroup F of k^* and an isomorphism

$H^2(G, k^*) \cong H^2_{abel}(G/G', F) \times H^2(G, W_k)$;

here G' denotes the commutator subgrup of G, cohomology is taken with respect to the trivial group action and $H^2_{abel}(G/G', F)$ denotes the subgroup of cocycle classes wich can be represented by symmetric cocycles. Hence, up to equivalence of cocycles, we may and do assume that there is a central symmetric cocycle $s': G/G' \times G/G' \to F$ and a central cocycle $t: G \times G \to W_k$ such that $f = s \cdot t$ where $s = inf_{G'}^G(s')$. Now Schur's lemma implies that there is a function $\alpha: G \to C^*$ such that $s = \delta\alpha$ (coboundary over C) and therefore $f = \delta\alpha \cdot t$. It follows that for every (simple) $C - f-$character $\chi: G \to C$ there is a (simple) $C - t-$character $\psi: G \to C$ such that $\chi = \alpha \cdot \psi$ and such that ψ belongs to a $C - t-$representation of G which can be lifted to a linear representation D of a finite central group extension $1 \to Z \to E \to G \to 1$ which is defined by t. If D is induced by a repesentation D_0 of a subgroup $H_0 \leq E$ such that $Z \leq H_0$, then ψ is induced by a $C - t-$character ψ_0 of the subgroup $H \coloneqq H_0/Z \leq G$ with respect to t, and the degree of ψ_0 is the degree of D_0. If G is nilpotent, then E is nilpotent. Assume that χ is simple. Then ψ is simple. Since finite nilpotent groups are monomial there is a subgroup $H \leq G$ and a function $\beta: H \to C^*$ such that $\delta\beta = t_{|H \times H}$ and $\psi = Ind_H^G(\beta)_t$. Hence from **(2)** we obtain $\chi = \alpha \cdot \psi = Ind_H^G(\alpha_{|H} \cdot \beta)_f$. This proves the following lemma.

https://doi.org/10.24355/dbbs.084-201810301515-0

(3) Lemma *If G is nilpotent then every simple $C-f-$character χ of G is monomial, i.e. there is a subgroup $H \leq G$ and also a function $\gamma\colon H \to C^*$ such that $\delta\gamma = f_{|H\times H}$ and $\chi = Ind_H^G(\gamma)_f$.*

We note

$$\gamma(x)^{m(x)} = \prod_{i=1}^{m(x)} f\left(x, x^i\right) = a_f(x)$$

for all $x \in H$ where $m(x)$ is the order of x. Hence $Ind_H^G(\gamma)_f$ is the $C-f-$character of a representation of G which is realizable in the subfield $L = k(\sqrt[m(x)]{a_f(x)} : x \in G)$ of C. Using Brauer's induction theorem [BT] or [SE], §10, 10.5, Thm. 19 or [LA[, XVIII, § 10, Thm. 15, and the twisted form **(2)** of the reciprocity law, a twisted form of Brauer's induction theorem was obtained in [O1], p. 584, and combining this result with **(3)** we obtain the following proposition.

(4) Proposition *For every $C-f-$ character χ of G there are nilpotent subgroups $H_1, \dots, H_r$ of G and functions $\gamma_1\colon H_1 \to C^*, \dots, \gamma_r\colon H_r \to C^*$ such that the coboundary $\delta\gamma_i$ is the restriction of f to $H_i \times H_i$ for all $i = 1, \dots, r$ and such that there are integers $n_1, \dots, n_r$ with the property*

$$\chi = \sum_{i=1}^{r} n_t Ind_{H_i}^G(\gamma_i)_f$$

We add that every $Ind_{H_i}^G(\gamma_i)_f$ is the character of a $C-f-$representation of G which is realizable over the field L. In order to complete the proof of **(1)** one argues as in the proof of the linear case $f = 1$, comp. e.g. [LA], XVIII, § 10, proof of Thm. 17: Decompose every $Ind_{H_i}^G(\gamma_i)_f$ in **(4)** as a sum of simple characters of $C-f-$representations of G which are realizable over L to obtain an expression of χ as a linear combination with nonnegative integer coefficients of simple characters which belong to simple representations of (C, G, f) which are realizable over L. This shows that χ itself belongs to a representation of (C, G, f) which is realizable over L and therefore completes the proof of **(1).**

Finally we discuss an example. The basic construction is taken from [O2], § 4; it makes use of relations between central 2-cocycles and bimultiplicative pairings which are explained in [Y], §2, and of elementary facts about Gauss sums. Let m be a positive integer >1 and let $W_m = \langle e^{2\pi i/m}\rangle \leq \mathbb{C}^*$ be the group of roots of unity of order m in $\mathbb{C}$. Assume that A is a finite abelian group of exponent m and that $t\colon A \times A \to W_m$ is a central 2-cocycle such that the associated symplectic pairing $\omega_t\colon A \times A \to W_m$, $\omega_t(x,y) \coloneqq t(x,y)/(t(y,x)$ for all $x, y \in A$, which is defined e.g. in [Y], §2, 2.1, (7), is nondegenerate. Le h denote a positive integer such that there is an epimorphism $G(\mathbb{Q}(e^{2\pi i/h})/\mathbb{Q}) \to A$. For every character χ of

https://doi.org/10.24355/dbbs.084-201810301515-0

A, viewed as a character of $G(\mathbb{Q}(e^{2\pi i/h})/\mathbb{Q})$, denote by $\tau(\chi)$ the corresponding Gauss sum, i.e.

$$\tau(\chi) := \sum_{a \bmod f(\chi)} \chi(a) e^{2\pi i a/f(\chi)}$$

where $f(\chi)$ is the conductor of χ ; for the terminology and elementary results on Gauss sums which are used here compare [LE], § 2. For every $x \in A$ let χ_x denote the character of A defined by $\chi_x(y) := \omega_t(x,y)$, $y \in A$, viewed as a character of $G(\mathbb{Q}(e^{2\pi i/h})/\mathbb{Q})$. Put $k := \mathbb{Q}(e^{2\pi i/m})$. Then $f: A \times A \to k^*$ defined by

$$f(x,y) := t(x,y)\tau(\chi_x)\tau(\chi_y)/\tau(\chi_x\chi_y),\ x,y \in A,$$

is a central 2-cocycle. Now we assume that the central 2-cocycle $t: A \times A \to \mu_m$ is a bimultiplicative pairing; see [Y], § 2, Thm. 2.2. Then the function $a_f: A \to k^*$ defined above is given by

$$a_f(x) = t(x,x)^{\frac{m(x)(m(x)+1)}{2}} \tau(\chi_x)^{m(x)},\ x \in A.$$

We have $\varepsilon(x) := t(x,x)^{\frac{m(x)(m(x)+1)}{2}} \in \{\pm 1\}$ for all $x \in A$, and $\alpha_f(x) = \sqrt[m(x)]{\varepsilon(x)} \cdot \tau(\chi_x)$, $x \in A$, is a root in $\mathbb{C}$ of order dividing $m(x)$ of $a_f(x)$. Hence the splitting field L of the twisted group algebra (k, A, f) from **(1)** is given by

$$L = k(\sqrt[m(x)]{\varepsilon(x)} \cdot \tau(\chi_x) : x \in A).$$

Denote by l the lcm of h and m. Since $\tau(\chi_x) \in \mathbb{Q}(e^{2\pi i/l})$ for every $x \in A$ we have $L \subset \mathbb{Q}(e^{\pi i/l})$. Especially $\mathbb{Q}(e^{\pi i/l})$ is a splitting field of the twisted group algebra (k, A, f).

References

[BR] R. Brauer: Über Systeme hyperkomplexer Zahlen, Math. Zeitschrift, 30, 1929, 79-107

https://doi.org/10.24355/dbbs.084-201810301515-0

[BT] R. Brauer, J. Tate: On the characters of finite groups, Annals of Mathematics, 62, 1955, 1-7

[CR] C. Curtis, I. Reiner: Representation theory of finite groups and associative algebras, Intersc. Publishers, New York, 1962

[LA] S. Lang: Algebra, Addison Wesley, New York, 1965

[LE] H. W. Leopoldt: Zur Geschlechtertheorie in abelschen Zahlkörpern, Math. Nachrichten, 9, 1953, 351-362

[O1] H. Opolka: Zerfällunskörper verschränkter Gruppenalgebren, J. of Algebra, 51, 1978, 579-596

[O2] H. Opolka: Cocycles, Galois theory and automorphic forms, parshin70.mi.ras.ru/materials.html

[SE] J. P. Serre: Représentations linéaires des groupes finis, deuxieme édition refondue, Hermann, Paris, 1971

[Y] K. Yamazaki: On projective representations and ring extensions of finite groups, Journal Fac. Sc. University of Tokyo, Section I 10,1964, 147-195

https://doi.org/10.24355/dbbs.084-201810301515-0